Microscopic theory of Anderson localization in a magnetic field

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(Received 20 October 1995; revised manuscript received 18 September 1996)

A microscopic diagrammatic Green's function theory is developed for the Anderson localization of a disordered two-dimensional electron gas under the influence of a weak external magnetic field. Using symmetry-adapted representations for the correlation functions, the influence of the magnetic field on the pole structure of the vertex function is investigated. A self-consistent effective potential is constructed from a class of diagrams, which are relevant if the quantum corrections to the conductivity are not completely suppressed by the magnetic field. Simple ladder diagrams do not play any significant role. The microscopic theory generalizes the self-consistent treatment derived by Vollhardt and Wölfle [in *Electronic Phase Transitions*, edited by W. Hanke and Y. V. Kopaev (Elsevier, Amsterdam, 1992); Phys. Rev. B **22**, 4666 (1980)] to the case of weak nonzero magnetic fields and justifies our former phenomenological approach to the problem. [S0163-1829(97)05304-X]

I. INTRODUCTION

At low temperatures the transport properties of microstructures are governed by quantum interference effects of conduction electrons during their elastic scattering on impurities. This gives rise to quantum corrections to the classical transport, which are due to singular backscattering described by ladder diagrams of the particle-particle channel and lead to anomalous temperature and magnetic-field dependences. Based on a one-parameter scaling theory of localization,[1] which clarified the role of quantum interference effects in microstructures, a theory of weak localization was established (see, e.g., Ref. 2), which holds in the limit of weak disorder, when the mean free path is much larger than the Fermi wavelength. The most important success of this theory was the explanation of the anomalous negative magnetoresistance (see, e.g., Ref. 3). Results of experiments were in excellent quantitative agreement with the weak localization theory so that magnetoresistance measurements have been used as a unique probe of electronic processes in such structures. At higher degrees of disorder, when the elastic scattering length λ is comparable with the Fermi wavelength ($k_F\lambda\sim 1$) the quantum corrections to the conductivity become even more relevant and a self-consistent description of disorder effects is desired.

Recently, the statistical properties of the energy spectrum of a three-dimensional (3D) disordered system in a magnetic field have been investigated.[4] Surprisingly, it has been found that there is a critical ensemble that is characteristic for the metal-insulator transition irrespective of the presence or absence of a magnetic field. The magnetic field actually changes the universality class, but, nevertheless, the behavior of the system at the critical point cannot be distinguished from the one without a magnetic field. From a field-theoretical point of view this result is completely unexpected. It has been speculated that the critical behavior is due to a new universality class that is independent of the presence or absence of time-reversal symmetry.[5–7] This astonishing result, which is the subject of some controversial discussions, is valid for 3D systems and does not apply to the 2D case, where one expects a phase transition only when a magnetic field is present. Furthermore, in two-dimensions topological invariants play a crucial role and a two–parameter scaling theory seems to be necessary. In high-quality quasi-two-dimensional electron-gas systems disorder scattering is intimately connected to transitions between various conducting and insulating states and their dependence on magnetic field. Such investigations are of fundamental interest in the theory of the quantized Hall effect. In this paper we do not treat this very interesting problem but restrict our consideration of Anderson localization to low magnetic fields, where the Landau orbit is much larger than the Fermi wavelength.

Finite-size scaling studies of a simple 2D one-electron model demonstrated that there are two phase transition points above a critical impurity concentration as a function of the magnetic-field strength.[8] The first transition point may occur at very low magnetic fields, whereas the second one is observed at a field strength that is about one order of magnitude smaller than the field at which the shrinkage of the wave function becomes dominant.

Vollhardt and Wölfle derived a self-consistent theory of Anderson localization by summing up the most strongly divergent diagrams. This theory had been successfully applied to both classical and quantum mechanical wave propagation phenomena in disordered systems (for a review see, e.g., Ref. 9). Several approaches have been proposed to generalize this self-consistent scheme to the case of broken time-reversal symmetry, as realized in the presence of an external magnetic field. The main problems of such a generalization are, on the one hand, the identification of the class of diagrams, which are to be treated self-consistently, and, on the other hand, to connect the approach with the correct perturbation theory for weak disorder. Due to the magnetic field the contribution of the maximally crossed diagrams of the Cooper channel is suppressed and localization can only be due to divergencies resulting from other diagrammatic contributions. Within an effective field theory a correction to the conductivity appears in two loop order. This contribution is independent of the magnetic field and results from the diffusion channel. In two dimensions there is a logarithmically divergent correction[2] that is weaker by a factor $(k_F\lambda)^{-1}$ compared to the zero-magnetic-field case but, nevertheless, dominates the behavior in sufficiently large systems. Diagrammatic Green's-function theories, which additionally incorporated particle-hole ladder diagrams,[10,11] were only of limited success and could not reproduce the field-theoretical results. The identification of the class of diagrams, which can be summed up in a self-consistent approximation and are most important in the vicinity of the metal-insulator phase transition point, is the main objective of a Green's-function theory of Anderson localization in a magnetic field. In this paper we propose such a microscopic approach that is valid if quantum corrections are not completely suppressed by the magnetic field and generalizes both the self-consistent theory of localization at vanishing magnetic field[12] and results of the weak localization theory with respect to the magnetoconductivity. Furthermore, the microscopic theory presented agrees with our previous studies based on a phenomenological reasoning.[13]

The paper is organized as follows. In Secs. II and III the symmetry properties of the Green's functions in a magnetic field are used to introduce the symmetry adapted Wigner representation. The influence of a weak magnetic field on the pole structure of the vertex function is investigated in Sec. IV and the Ward identity is discussed in Sec. V. The main aim of our paper, namely, the selection of diagrams, that have to be treated self-consistently in order to construct an adequate picture of Anderson localization in a magnetic field, is presented in Sec. VI. We close with a summary in Sec. VII.

II. DYSON EQUATION IN WIGNER REPRESENTATION

To consider the peculiarities of the impurity scattering in a 2D electron gas under the influence of a transverse magnetic field we profit from symmetry-adapted coordinates. If an external magnetic field is applied to the system, translational invariance no longer exists so that a simple Fourier transformation does not reduce the number of independent variables. Consequently, the Fourier-transformed impurity-averaged one-particle Green's function, for instance, still depends on two independent momenta $\boldsymbol{k}_1$and $\boldsymbol{k}_2$. Nevertheless, one may exploit the following symmetry property of the averaged one-particle Green's function:

$$\mathcal{G}(\mathbf{r}_1,\mathbf{r}_2,z)=\mathcal{G}(\mathbf{r}_1+\mathbf{r},\mathbf{r}_2+\mathbf{r},z)\exp[i\mathbf{A}(\mathbf{r})(\mathbf{r}_1-\mathbf{r}_2)], \quad (1)$$

which allows the introduction of a symmetry adapted representation. Here $\mathbf{A}(\mathbf{r})$ is the vector potential of the magnetic field in the symmetric gauge and z is on the imaginary time axis. Performing a Fourier transformation and introducing the new wave vectors $\boldsymbol{k},\boldsymbol{\kappa}$ by $\boldsymbol{k}_1=\boldsymbol{k}+\boldsymbol{\kappa}/2$ and $\boldsymbol{k}_2=\boldsymbol{k}-\boldsymbol{\kappa}/2$, the symmetry relation Eq. (1) may be expressed by

$$\mathcal{G}(\boldsymbol{k},\boldsymbol{\kappa},z)=\mathcal{G}(\boldsymbol{k}-\boldsymbol{A}(\boldsymbol{r}),\boldsymbol{\kappa},z)e^{i\boldsymbol{\kappa}\cdot\mathbf{r}} \quad (2)$$

which indicates that it is useful to introduce the so-called Wigner transformed Green's function, depending only on one wave vector[14]

$$\mathcal{G}(\boldsymbol{k},z)=\sum_{\boldsymbol{\kappa}}\mathcal{G}(\boldsymbol{k}+\boldsymbol{A}(\boldsymbol{r}),\boldsymbol{\kappa},z)e^{-i\boldsymbol{\kappa}\cdot\mathbf{r}}. \quad (3)$$

This symmetry-adapted representation of the Green's function accounts for the invariance of the system under magnetic translations. The inverse transformation is simply given by

$$\mathcal{G}(\boldsymbol{k},\boldsymbol{\kappa},z)=\int d\boldsymbol{r}\mathcal{G}(\boldsymbol{k}-\boldsymbol{A}(\boldsymbol{r}),z)e^{i\boldsymbol{\kappa}\cdot\mathbf{r}}. \quad (4)$$

Now it is straightforward to see that the Fourier-transformed Dyson equation with the dispersion relation $\varepsilon(\boldsymbol{k})$ of the underlying lattice

$$\{z-\varepsilon[\boldsymbol{k}_1+\boldsymbol{A}(i\nabla_{\boldsymbol{k}_1})]\}\mathcal{G}(\boldsymbol{k}_1,\boldsymbol{k}_2,z)=\delta(\boldsymbol{k}_1-\boldsymbol{k}_2)$$
$$+\sum_{\overline{\boldsymbol{k}}}\Sigma(\boldsymbol{k}_1,\overline{\boldsymbol{k}},z)\mathcal{G}(\overline{\boldsymbol{k}},\boldsymbol{k}_2,z) \quad (5)$$

simplifies considerably if one introduces the Wigner transformed Green's function (3). To that end we introduce the new variables $\boldsymbol{k},\boldsymbol{\kappa}$ and sum over $\boldsymbol{\kappa}$. Furthermore, we take into account that in the symmetric gauge the equation $\boldsymbol{A}(\boldsymbol{r})\boldsymbol{r}=0$ is fulfilled. Throughout we assume that the self-energy Σ is independent of momentum. Then, in the symmetry-adapted Wigner representation the Dyson equation has the form[15]

$$\{z-\varepsilon[\boldsymbol{k}+\boldsymbol{A}(i\nabla_{\boldsymbol{k}})]\}\mathcal{G}(\boldsymbol{k},z)=1+\Sigma(z)\mathcal{G}(\boldsymbol{k},z), \quad (6)$$

which has the following explicit solution for free electrons in the conduction band $\varepsilon(\boldsymbol{k})=\hbar k^2/2m^*$:[16]

$$\mathcal{G}(k,z)=2e^{-(kl)^2}\sum_{n=0}^{\infty}\frac{(-1)^n L_n[2(kl)^2]}{z-\hbar\omega_c(n+1/2)-\Sigma(z)}. \quad (7)$$

Here ω_c is the cyclotron frequency, $l=\sqrt{\hbar/m^*\omega_c}$ is the magnetic length, and L_n are Laguerre polynomials. We want to point out that the Dyson equation (6) parallels as closely as possible the one that is obtained in the absence of any magnetic field.

III. WIGNER REPRESENTATION OF TWO-PARTICLE FUNCTIONS

In the one-electron model considered both the conductivity and the dynamical diffusion coefficient are expressed by the density-density correlation function

$$\Phi(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3,\boldsymbol{r}_4;z,z')=\langle G(\boldsymbol{r}_1,\boldsymbol{r}_3,z)G(\boldsymbol{r}_4,\boldsymbol{r}_2,z')\rangle. \quad (8)$$

In accordance with Eq. (1) and as a consequence of the invariance against magnetic translations, Φ exhibits the following symmetry property in the site representation:

$$\Phi(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3,\boldsymbol{r}_4;z,z')=\Phi(\boldsymbol{r}_1+\boldsymbol{r},\boldsymbol{r}_2+\boldsymbol{r},\boldsymbol{r}_3+\boldsymbol{r},\boldsymbol{r}_4+\boldsymbol{r};z,z')$$
$$\times\exp[i\boldsymbol{A}(\boldsymbol{r})(\boldsymbol{r}_1-\boldsymbol{r}_2-\boldsymbol{r}_3+\boldsymbol{r}_4)]. \quad (9)$$

As in Sec. II, a direct Fourier transformation does not result in any simplification because there is no momentum conser-

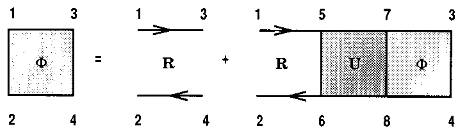

FIG. 1. Diagrammatic representation of the Bethe-Salpeter equation.

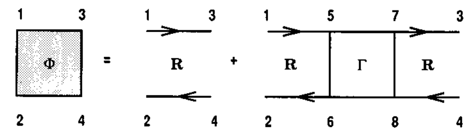

FIG. 2. Diagrammatic introduction of the vertex function Γ.

vation and still four variables $\boldsymbol{k}_1, \ldots, \boldsymbol{k}_4$ appear. Therefore, we introduce new independent quantities by

$$\boldsymbol{k}_1=\boldsymbol{k}-\frac{\boldsymbol{\kappa}+\boldsymbol{\eta}}{2}, \quad \boldsymbol{k}_3=\boldsymbol{k}'-\frac{\boldsymbol{\kappa}}{2}$$

$$\boldsymbol{k}_2=\boldsymbol{k}+\frac{\boldsymbol{\kappa}+\boldsymbol{\eta}}{2}, \quad \boldsymbol{k}_4=\boldsymbol{k}'+\frac{\boldsymbol{\kappa}}{2}, \tag{10}$$

so that Eq. (9) is transformed into

$$\Phi(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{\kappa},\boldsymbol{\eta};z,z')=\Phi(\boldsymbol{k}+\boldsymbol{A}(\boldsymbol{r}),\boldsymbol{k}'+\boldsymbol{A}(\boldsymbol{r}),\boldsymbol{\kappa},\boldsymbol{\eta};z,z')e^{i\boldsymbol{\eta}\cdot\boldsymbol{r}}. \tag{11}$$

Now the Wigner representation of the density-density correlation function can be introduced, thereby making it possible to reduce the number of independent variables

$$\Phi(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{\kappa};z,z')=\sum_{\boldsymbol{\kappa}',\boldsymbol{\eta}}\int d\boldsymbol{r}e^{i(\boldsymbol{\kappa}-\boldsymbol{\kappa}')\boldsymbol{r}} \times\Phi(\boldsymbol{k},\boldsymbol{k}'+\boldsymbol{A}(\boldsymbol{r}),\boldsymbol{\kappa}',\boldsymbol{\eta};z,z'). \tag{12}$$

The inverse transformation has the form

$$\Phi(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{\kappa},\boldsymbol{\eta};z,z')=\sum_{\boldsymbol{\kappa}'}\int d\boldsymbol{r}d\boldsymbol{r}'e^{i(\boldsymbol{\kappa}'-\boldsymbol{\kappa})\boldsymbol{r}'}e^{i\boldsymbol{\eta}\cdot\boldsymbol{r}}\Phi(\boldsymbol{k}+\boldsymbol{A}(\boldsymbol{r}),\boldsymbol{k}'+\boldsymbol{A}(\boldsymbol{r}+\boldsymbol{r}'),\boldsymbol{\kappa}';z,z'). \tag{13}$$

Introducing the Wigner transformed function Eq. (12) has the advantage that the correlation functions depend only on three vectors $\boldsymbol{k},\boldsymbol{k}'$, and $\boldsymbol{\kappa}$ as in the homogeneous case $(\boldsymbol{A}=\boldsymbol{0})$, where quasimomentum conservation holds.

IV. CALCULATION OF THE VERTEX FUNCTION

The two-particle function Φ satisfies the Bethe-Salpeter equation, which is diagrammatically depicted in Fig. 1. Our aim is to consider the influence of a magnetic field on Anderson localization within a self-consistent effective potential approach. Because in the weak-scattering limit the irreducible vertex U is independent of the momenta it is expedient to start from the so-called diagonal vertex approximation.[17] In this approximation the Bethe-Salpeter equation is easily solved in the Wigner representation and one obtains[15]

$$\Phi(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{\kappa};z,z')=R(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{\kappa};z,z')+\frac{u(z,z')\Sigma_{\boldsymbol{k}_1}R(\boldsymbol{k}_1,\boldsymbol{k}',\boldsymbol{\kappa};z,z')\Sigma_{\boldsymbol{k}_2}R(\boldsymbol{k},\boldsymbol{k}_2,\boldsymbol{\kappa};z,z')}{1-u(z,z')\Sigma_{\boldsymbol{k}_1,\boldsymbol{k}_2}R(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{\kappa};z,z')}, \tag{14}$$

where R is the Wigner transform of the product of two one-particle Green's functions ($\mathcal{G}\mathcal{G}$) and u is the effective potential. To proceed further one may express the density-density correlation function Φ by the vertex function Γ as shown in Fig. 2. Using the Wigner representation introduced in Sec. III, this equation simplifies considerably and takes the form

$$\Phi(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{\kappa};z,z')=R(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{\kappa};z,z')+\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2}R(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{\kappa};z,z') \times\Gamma(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{\kappa};z,z')R(\boldsymbol{k}_2,\boldsymbol{k}',\boldsymbol{\kappa};z,z'). \tag{15}$$

From Eqs. (14) and (15) it is seen that the vertex function Γ does not depend on $\boldsymbol{k}$ and $\boldsymbol{k}'$. Therefore, we obtain from these equations

$$\Gamma(\boldsymbol{\kappa};z,z')=\frac{u(z,z')}{1-u(z,z')\Sigma_{\boldsymbol{k}}\mathcal{G}(\boldsymbol{k}+\boldsymbol{\kappa}/2,z)\mathcal{G}(\boldsymbol{k}-\boldsymbol{\kappa}/2,z')}. \tag{16}$$

Now the pole structure of the vertex function has to be investigated, which requires an analytic continuation of the one-particle Green's functions in the denominator of Eq. (16) $(z\rightarrow E+\hbar\omega+i\varepsilon$ and $z'\rightarrow E-i\varepsilon)$. For the product $\mathcal{G}^r\mathcal{G}^a$ a differential equation is easily derived from the identity

$$[(\mathcal{G}^r)^{-1}-(\mathcal{G}^a)^{-1}]\mathcal{G}^r\mathcal{G}^a=\mathcal{G}^a-\mathcal{G}^r \tag{17}$$

by considering the Dyson equation (6) and by transferring the derivatives on $\boldsymbol{k}$ to $\boldsymbol{\kappa}$,

$$\begin{aligned}&\left(\hbar\omega+\frac{i\hbar}{\tau(E,\omega)}\right)\mathcal{G}^r\left(\boldsymbol{k}+\frac{\boldsymbol{\kappa}}{2},E+\hbar\omega\right)\mathcal{G}^a\left(\boldsymbol{k}-\frac{\boldsymbol{\kappa}}{2},E\right)\\&\quad+\mathcal{G}^a\left(\boldsymbol{k}-\frac{\boldsymbol{\kappa}}{2},E\right)\varepsilon\left(\boldsymbol{k}+2\boldsymbol{A}(i\nabla_{\boldsymbol{\kappa}})+\frac{\boldsymbol{\kappa}}{2}\right)\mathcal{G}^r\left(\boldsymbol{k}+\frac{\boldsymbol{\kappa}}{2},E+\hbar\omega\right)\\&\quad-\mathcal{G}^r\left(\boldsymbol{k}+\frac{\boldsymbol{\kappa}}{2},E+\hbar\omega\right)\varepsilon\left(\boldsymbol{k}-2\boldsymbol{A}(i\nabla_{\boldsymbol{\kappa}})-\frac{\boldsymbol{\kappa}}{2}\right)\\&\quad\times\mathcal{G}^a\left(\boldsymbol{k}-\frac{\boldsymbol{\kappa}}{2},E\right)\\&\quad=2\pi iN(\boldsymbol{k},\boldsymbol{\kappa},E,\omega).\end{aligned} \tag{18}$$

Here the self-energy difference is expressed by an effective scattering time τ according to

$$\Sigma^r(E+\hbar\omega)-\Sigma^a(E)=-\frac{i\hbar}{\tau(E,\omega)}. \tag{19}$$

We are interested in the limit $\boldsymbol{\kappa},\omega\to 0$ and consider $(\boldsymbol{\kappa},\omega)$ corrections resulting from the pole in Γ so that the $(\boldsymbol{\kappa},\omega)$ dependence on the right-hand side of Eq. (18) can be omitted and $N(\boldsymbol{k},\boldsymbol{\kappa},E,\omega)$ is replaced by

$$N(\boldsymbol{k},E)=\frac{1}{2\pi i}[\mathcal{G}^a(\boldsymbol{k},E)-\mathcal{G}^r(\boldsymbol{k},E)]. \tag{20}$$

Furthermore, we restrict our consideration to the region of weak magnetic fields and treat only its influence on the pole structure, which results in logarithmic corrections to the dynamical diffusion coefficient. In this approximation Eq. (18) may be further simplified with the result

$$\left(\hbar\omega-\nabla_{\boldsymbol{k}}\varepsilon(\boldsymbol{k})\hat{p}(\boldsymbol{\kappa})+\frac{i\hbar}{\tau(E,\omega)}\right)\mathcal{G}^r\left(\boldsymbol{k}+\frac{\boldsymbol{\kappa}}{2},E+\hbar\omega\right)\times\mathcal{G}^a\left(\boldsymbol{k}-\frac{\boldsymbol{\kappa}}{2},E\right)=2\pi iN(\boldsymbol{k},E), \tag{21}$$

where the following operator was introduced:

$$\hat{p}(\boldsymbol{\kappa})=\boldsymbol{\kappa}+2\boldsymbol{A}(i\nabla_{\boldsymbol{\kappa}}). \tag{22}$$

It is seen from Eq. (22) that the charge of the related cooperon is $2e$, which has been accounted for in weak-localization theory too. Now an iteration of Eq. (21) up to the third order in $\hbar\omega,\boldsymbol{\kappa}$, and $\boldsymbol{A}$ together with Eq. (16) gives our final result for the vertex function

$$\Gamma(\boldsymbol{\kappa},E,\omega)=\frac{u(E,\omega)}{\tau(E,\omega)}\left[-i\omega+\sum_{i,j}\hat{p}_i(\boldsymbol{\kappa})\hat{p}_j(\boldsymbol{\kappa})D_{ij}(E,\omega)\right]^{-1}, \tag{23}$$

where the spectral diffusion coefficient is defined by

$$D_{ij}(E,\omega)=\frac{\tau(E,\omega)}{\hbar^2}\frac{1}{N(E)}\sum_{\boldsymbol{k}}N(\boldsymbol{k},E)\frac{\partial\varepsilon(\boldsymbol{k})}{\partial k_i}\frac{\partial\varepsilon(\boldsymbol{k})}{\partial k_j}. \tag{24}$$

$N(E)=\Sigma_{\boldsymbol{k}}N(\boldsymbol{k},E)$ is the density of states at the Fermi level. The vertex function Γ in Eq. (23) corresponds to the Cooper propagator, which we introduced in our former phenomenological treatment of the problem.[13]

V. WARD IDENTITY

The consideration of the due relation between one- and two-particle properties via the Ward identity is inevitable in the microscopic approach to Anderson localization. From the definition of the density-density correlation function Φ in Eq. (8) the following equation is easily derived for the Wigner transformed correlation functions:

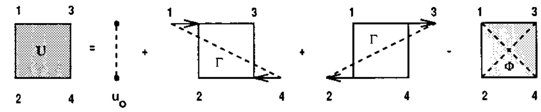

FIG. 3. Diagrammatic representation of the equation for the irreducible vertex.

$$\sum_{\boldsymbol{k}'}\Phi(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{0};E,\omega)=-\frac{\mathcal{G}^r(\boldsymbol{k},E+\hbar\omega)-\mathcal{G}^a(\boldsymbol{k},E)}{\hbar\omega}. \tag{25}$$

After summing up this equation over $\boldsymbol{k}$ and exploiting the explicit solution for Φ from Eq. (14) one obtains

$$u(E,\omega)=\frac{1}{R(E,\omega)}-\frac{\hbar\omega}{2\pi iN(E)}, \tag{26}$$

where

$$R(E,\omega)=\sum_{\boldsymbol{k},\boldsymbol{k}'}R(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{0};E,\omega). \tag{27}$$

A second equation for $R(E,\omega)$ can be easily derived by following the same steps as put forth in Sec. IV and by setting there $\boldsymbol{\kappa}=\boldsymbol{0}$. Consequently, from Eq. (21) we easily find

$$R(E,\omega)=\frac{2\pi iN(E)}{\hbar\omega+i\hbar/\tau(E,\omega)}, \tag{28}$$

which together with Eq. (26) results in

$$u(E,\omega)=\frac{\hbar}{2\pi N(E)\tau(E,\omega)}. \tag{29}$$

This equation relates the up to now unknown effective potential u with the scattering time τ and is an essential connecting link in our self-consistent scheme.

VI. EFFECTIVE POTENTIAL

There is still the challenging problem to be addressed, namely, to select diagrams for the irreducible vertex U, from which the effective potential u is constructed. The class of diagrams has to be chosen so as to ensure that in the limit of vanishing magnetic fields the theory by Vollhardt and Wölfle[9] is recovered. The problem of finding appropriate diagrammatic contributions is complicated by the observation that simple ladder diagrams of the scattering channels are unimportant. We consider the same set of diagrams, which we already introduced in our study of Anderson localization in an external electric field[18] and is depicted in Fig. 3. Among other and more essential contributions both the second and the third term on the right-hand side of Fig. 3 generate maximally crossed diagrams of the Cooper channel. This double counting is corrected by the fourth diagram, whose contribution, however, can be neglected if the renormalized coupling is much larger than the bare coupling constant u_0. This result is in line with the expectation that in the vicinity of the magnetic-field-mediated metal-insulator phase transition the maximally crossed ladder diagrams of the weak-localization theory do not play any significant role. The remaining part of the equation for the irreducible vertex has the following form in the Wigner representation

$$U(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{0};E,\omega)$$
$$=u_0+u_0\sum_{\boldsymbol{\kappa}} \mathcal{G}^r(\boldsymbol{k}+\boldsymbol{\kappa},E+\hbar\omega)\mathcal{G}^a(\boldsymbol{k}'-\boldsymbol{\kappa},E)\Gamma(\boldsymbol{\kappa};E,\omega)$$
$$+u_0\sum_{\boldsymbol{\kappa}} \mathcal{G}^r(\boldsymbol{k}'+\boldsymbol{\kappa},E+\hbar\omega)\mathcal{G}^a(\boldsymbol{k}-\boldsymbol{\kappa},E)\Gamma(\boldsymbol{\kappa};E,\omega), \tag{30}$$

where $u_0=\hbar/[2\pi N(E)\tau_0]$ and τ_0 is the bare elastic scattering time. The effective coupling constant u may be obtained from the irreducible kernel U by means of an average over the Fermi surface

$$u(E,\omega)=\frac{1}{N(E)^2}\sum_{\boldsymbol{k},\boldsymbol{k}'} U(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{0};E,\omega)N(\boldsymbol{k},E)N(\boldsymbol{k}',E). \tag{31}$$

Because a singular $\boldsymbol{\kappa}$ dependence is introduced only by the vertex function Γ, Eq. (30) may be further simplified by averaging it over the Fermi surface, which we denote simply by a bar

$$u(E,\omega)=u_0\left[1-2\,\overline{\mathcal{G}^r(\boldsymbol{k},E+\hbar\omega)\mathcal{G}^a(\boldsymbol{k}',E)}\sum_{\boldsymbol{\kappa}} \Gamma(\boldsymbol{\kappa};E,\omega)\right]. \tag{32}$$

Only terms that are proportional to $\mathcal{G}^r\mathcal{G}^a$ contribute significantly to the average so that we obtain

$$\overline{\mathcal{G}^r\mathcal{G}^a}\cong-\frac{1}{4\pi^2}\left(\frac{R(E,\omega)}{N(E)}\right)^2=-\frac{1}{4\pi^2N(E)^2u(E,\omega)^2}. \tag{33}$$

Inserting this result into Eq. (32), the following relation between the effective potential u and the vertex function Γ is obtained:

$$u(E,\omega)=u_0\left[1+\frac{\Sigma_{\boldsymbol{\kappa}}\Gamma(\boldsymbol{\kappa};E,\omega)}{2\pi^2N(E)^2u(E,\omega)^2}\right]. \tag{34}$$

For an isotropic system we introduce the renormalized dynamical diffusion coefficient

$$D(E,\omega)=v_F^2\tau(E,\omega)/d, \tag{35}$$

which is in accordance with Eq. (24). d denotes the dimension of the lattice. Considering the result (29) derived from the Ward identity, Eq. (34) for the self-consistent potential takes the following form for the diffusion coefficient

$$D(E,\omega)=\frac{D_0}{1+\dfrac{1}{\hbar\pi N(E)}\sum_{\boldsymbol{\kappa}} C(\boldsymbol{\kappa};E,\omega)}, \tag{36}$$

where $D_0=v_F^2\tau_0/d$. As in our former phenomenological approach,[13] the autocorrelation function $\Sigma_{\boldsymbol{\kappa}}C(\boldsymbol{\kappa};E,\omega)$ of the Cooper propagator has been introduced, the Fourier components of which satisfy the following differential equation according to Eq. (23)

$$\{-i\omega+D(E,\omega)\hat{p}(\boldsymbol{\kappa})^2\}C(\boldsymbol{\kappa};E,\omega)=1\,. \tag{37}$$

At the absence of any magnetic field the operator $\hat{p}(\boldsymbol{\kappa})$ is simply given by $\boldsymbol{\kappa}$ so that Eq. (36), together with Eq. (37), reproduces the self-consistent equations for the dynamical diffusion coefficient worked out by Vollardt and Wölfle (for a review and a discussion of this equation see Ref. 9). When electrons are subject to a weak external magnetic field Eq. (37) is conveniently transformed back to the site representation, which leads to the equation

$$\{-i\omega-D(E,\omega)[\nabla_{\boldsymbol{r}}-i2\boldsymbol{A}(\boldsymbol{r})]^2\}C(\boldsymbol{r};E,\omega)=\delta(\boldsymbol{r}). \tag{38}$$

For the special case of a disordered two-dimensional system the following self-consistent equation for the dynamical diffusion coefficient results from Eqs. (36) and (38):

$$D=D_0-\frac{1}{2\pi^2\hbar N_F}\left[\psi\left(\frac{1}{2}+\left\{\frac{lk_0}{2}\right\}^2+s\frac{l^2}{4D}\right)-\psi\left(\frac{1}{2}+s\frac{l^2}{4D}\right)\right], \tag{39}$$

where $l=\sqrt{\hbar c/eH}$ is the magnetic length, $k_0\sim 1/v_F\tau_0$ an appropriate momentum cutoff, ψ the digamma function, and s $(-i\omega\rightarrow s)$ the variable of the Laplace transformation, which can be identified with the inverse inelastic scattering time $1/\tau_\varepsilon$. Solutions of this transcendental equation for the diffusion coefficient were compared with results of other self-consistent approaches and with experimental data in Ref. 13. In the weak-coupling limit $(k_F\lambda\gg 1)$, where the disorder can be treated within the framework of perturbation theory, the diffusion coefficient D on the right-hand side of Eq. (39) can be replaced by D_0, which leads to the well-established weak-localization theory of the magnetoconductivity.[3] Our basic results [Eqs. (36)–(38)], which we derived in this paper on the basis of a microscopic model, have already been used to thoroughly investigate Anderson localization in anisotropic three-dimensional electron systems under the influence of weak magnetic fields.[19] There it has been demonstrated that there is a metal-insulator phase boundary, which separates localized states at low-magnetic-field strengths, where the diffusion coefficient scales to zero, from extended states at somewhat higher magnetic fields. This can be seen immediately from Eq. (39) in the limit of vanishing inelastic scattering $(s\rightarrow 0)$ and weak magnetic fields, where we obtain [cf. Eq. (22) in Ref. 19]

$$\frac{D}{D_0}=1+\frac{1}{\pi}\frac{1}{k_F\lambda_0}\{\ln[(k_F\lambda_0)(\omega_c\tau_0)]-C\}, \tag{40}$$

with $\omega_c = \hbar/m^* l^2$ being the cyclotron frequency and C Euler's constant. From Eq. (40) it follows that there is a critical magnetic field determined by

$$\omega_c^* \tau_0 = \frac{e^C}{k_F \lambda_0} \exp(-\pi k_F \lambda_0), \qquad (41)$$

at which the renormalized diffusion coefficient vanishes. Below this magnetic-field strength the states are localized. This is due to the fact that one obtains two solutions in this magnetic-field region if the infrared cutoff vanishes ($s=0$). The physically relevant one is simply $D=0$, which is approximated by the numerical solution of Eq. (39) when s goes to zero. The localization length of the states in weak magnetic fields is, however, much larger than the zero-field localization length.

From a renormalization-group analysis[20,21] it is known that in $d=2$ dimensions logarithmically divergent corrections exist that are due to diffusion poles and dominate the magnetic-field-mediated localization if the system size L is much larger than the zero-field localization length ξ. This scale-invariant contribution that does not depend on the magnetic field is smaller by a factor $(k_F\lambda_0)^{-1}$ compared to the suppressed quantum corrections resulting from the Cooper channel. Nevertheless, for sufficiently large system sizes and inelastic scattering times only this classical localizing term survives. Although our self-consistent Green's-function approach includes particle-hole ladder diagrams via the Bethe-Salpeter equation we did not treat a renormalization of the diffusions from which such a result could be derived. Rather we focused our attention on the magnetic-field-induced renormalization of the quantum corrections that are relevant if the magnetic length l is larger than the system size L or the inelastic scattering length. That means that we restricted our treatment to the case where the cooperons are not completely suppressed by the magnetic field and constructed an effective potential from the particle-particle channel. Consequently, our approach is applicable in the weak-magnetic-field region ($l>L$) where it approaches the weak-localization theory of the magnetoresistance if the disorder becomes weak ($k_F\lambda_0 \gg 1$). Up to now it has not been clear how magnetic-field-induced electron localization due to divergencies coming from particle-hole diffusion poles could be described within the framework of a self-consistent diagrammatic theory.

For a three-dimensional system we calculate the localization length ξ_B in a magnetic field. Near the metal-insulator transition point Eqs. (36) and (38) can be cast into the closed form[19]

$$\frac{2h}{\pi k_F \lambda_0} \int_0^1 dx \int_0^\infty dt \frac{\sinh(t/2)}{\sinh(ht)} \times \exp\{-[1/2 + x^2 + 1/(\xi_B k_0)^2]t\} = 1, \qquad (42)$$

where $h = 2/(lk_0)^2$ is the magnetic-field parameter. Solving this equation in the magnetic-field region considered ($h \ll 1$), we obtain the same critical exponent $\nu = -1$ for the divergence of the localization length ξ_B as in the case without any magnetic field.[9] This surprising result is in line with conclusions drawn from numerical studies.[4] If the numerical results are indeed valid for large system sizes satisfying $l<L$ this agreement demonstrates that the particle-hole renormalization does not play any significant role at least near the 3D metal-insulator transition point. Further work seems to be necessary to decide the question whether the numerical data really apply to an infinite system or are restricted to system sizes smaller than the magnetic length.

Our self-consistent model has been used to investigate magnetic-field-mediated Anderson localization in anisotropic systems[19] at an arbitrary alignment of the magnetic field.[22] There it has been shown that the approach is in accord with numerous well-established limiting results.

VII. SUMMARY

The asymptotic critical behavior of the disordered system in a magnetic field is governed by the unitary universality class that reflects effectively broken time-reversal symmetry if $l<L$. On the other hand, under the condition $L \lesssim l$ most of the relevant electronic trajectories cover the whole area and the system behaves approximately according to the orthogonal symmetry.[23] In this case the cooperon is not completely suppressed by the magnetic field and the construction of a self-consistent theory of Anderson localization that includes an external magnetic field is straightforward. Using a microscopic Green's-function approach we identified a class of vertex diagrams that accounts for the magnetic-field-induced metal-insulator phase transition in a disordered two-dimensional electron gas if $L \lesssim l$. The basic ideas we presented in this paper have already been exploited to treat Anderson localization in an external electric field.[18] It has been shown that the particle-hole and particle-particle ladder diagrams do not contribute significantly and that the diagrams, which have to be summed up, exhibit a complicated structure. The selected diagrammatic contributions have been used to construct a self-consistent effective potential that is connected with the dynamical diffusion coefficient via the Ward identity. We restricted our consideration to the lowest-order change of the pole structure in the vertex function due to a magnetic field, allowing us to determine logarithmic corrections to the renormalized diffusion coefficient. The second-order contribution, which would introduce the Hall component of the diffusion tensor, has been neglected. The calculation greatly profited from the introduction of symmetry adapted representations.

Our basic results completely agree with the former phenomenological approach to the problem,[13] which has already been used to treat the field dependence of Anderson localization in anisotropic systems[19] and at arbitrary orientations of the magnetic field.[22] The microscopic Green's-function approach that we presented here generalizes the self-consistent theory worked out by Vollhardt and Wölfle[9] and reproduces the magnetoconductivity results of the weak-localization theory in the limit of weak disorder.

Further progress may be expected from an extension of our microscopic description to higher magnetic fields, where the quantum Hall effect becomes relevant. But it is also suggestive to assume that a completely new approach has to be envisaged in this case because in the region of high magnetic fields almost all states are strongly localized.

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