Multifractal dimension of chaotic attractors in a driven semiconductor superlattice

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The multifractal dimension of chaotic attractors has been studied in a weakly coupled superlattice driven by an incommensurate sinusoidal voltage as a function of the driving voltage amplitude. The derived multifractal dimension for the observed bifurcation sequence shows different characteristics for chaotic, quasiperiodic, and frequency-locked attractors. In the chaotic regime, strange attractors are observed. Even in the quasiperiodic regime, attractors with a certain degree of strangeness may exist. From the observed multifractal dimensions, the deterministic nature of the chaotic oscillations is clearly identified. [S0163-1829(99)03032-5]

I. INTRODUCTION

Dissipative dynamical systems (e.g., current oscillations in a solid-state system driven by an ac voltage) are characterized by the attraction of all trajectories passing through a certain domain of phase space towards a geometrical object called an ''attractor.''[1,2] The attractor can be a single point, a number of isolated points, or an extended object in phase space. Therefore, the structure of the attractor is crucial for the understanding of the dynamical behavior of a system.[1,2] In a physical experiment, one usually employs the so-called first return maps or Poincaré maps to visualize and analyze the underlying attractors.[1,2] For example, for frequency locking, which corresponds to periodic oscillations, the attractor consists of a set of discrete points so that the dimension of the attractor is zero. For quasiperiodic oscillations, the Poincaré map usually consists of a loop with a homogeneous point density so that the dimension of the attractor is one. In the chaotic regime, the structure of the attractor is called ''strange.'' The dimension of the attractor becomes now a fractal instead of an integer number. Strange attractors are usually characterized by a multifractal dimension D_q, which is a measure of the strangeness of the attractor.[3,4]

Resonant tunneling in weakly coupled semiconductor superlattices (SL's) results in a negative differential velocity (NDV) in the drift velocity vs electric-field characteristics.[5–7] Due to the presence of the NDV, the vertical transport in the SL's exhibits nonlinear properties, which result in such phenomena as domain formation,[8] multistability,[9] and self-sustained oscillations.[6,7,10] In this nonlinear system, chaos has also been studied theoretically[11] and experimentally.[12,13] Recently, an explosive bifurcation to chaos has been experimentally observed in the power spectra of a driven SL and unambiguously identified using Poincaré maps.[13] During the bifurcation sequence from frequency-locking to chaos, the Poincaré map expands from isolated points to connected branches, indicating that the dimension of the underlying attractors increases with increasing amplitude of the driving voltage. When the system enters the chaotic regime, the Poincaré map becomes more complex indicating that the attractor acquires a higher dimension. In order to prove the increase of the attractor dimension and to obtain additional information about the underlying chaos, it is necessary to quantitatively characterize the dimension of the attractors as a function of the external driving voltage amplitude.

In this paper we will report on the multifractal dimension of the attractors for a weakly coupled SL under an external driving voltage. The obtained fractal dimensions unambiguously identify the presence of strange attractors and the deterministic nature of the chaotic oscillations. We also find that strange attractors may exist in the quasiperiodic region.

The paper is organized as follows. In Sec. II the experimental bifurcation route to chaos in a driven SL is briefly reviewed. In Sec. III the dimension of the attractors in the observed bifurcation sequence is derived from the experimental Poincaré maps and discussed. Finally, a short summary is given in Sec. IV.

II. EXPERIMENTAL FACTS

In the experiments the dc bias is fixed and the sample is driven by an incommensurate ac voltage. The detailed information about the samples and the performed experiments can be found in Ref. 13. In this paper we will discuss the bifurcation route at the dc bias of 6.574 V, where the current self-oscillations exhibit an intrinsic fundamental frequency f_0 of 30.5 MHz. In the experiment, the driving frequency (f_d) is set to the golden mean $(1+\sqrt{5})/2 \approx 1.618$ times f_0, i.e., 49.4 MHz, while the driving amplitude V_{ac} is varied. The observed power spectra as a function of V_{ac} is given in Fig. 1(a) in Ref. 13, which shows the following sequence: quasiperiodicity → synchronized chaos → frequency locking → chaos → chaos with higher complexity. The Poincaré maps derived from real-time traces further verify this bifurcation route to chaos.[13] At small V_{ac}, such as 9 and 34 mV shown in Fig. 3 of Ref. 13, the Poincaré map consists of a

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closed loop, corresponding to a quasiperiodic attractor. At 39 mV, the loop is split into three extended branches, indicating the appearance of synchronized chaos. By increasing V_{ac} from 39 to 40 mV, the system enters a frequency-locked state with a Poincaré map consisting of only three points (they are broadened due to noise). This frequency-locked state exists up to values of V_{ac} of about 50 mV, while at 52 mV two of these three points expand into branches demonstrating the onset of chaos. Within this chaotic regime, a number of chaotic attractors with different complexity are revealed. The Poincaré maps between 50 and 56 mV show that the bifurcation from frequency-locking to chaos is explosive, i.e., it is accompanied by an abrupt increase of the attractor dimension (from three points with zero dimension to an object with finite dimensions).

In Sec. III we will estimate the multifractal dimension of the attractors from the experimental Poincaré maps in order to analyze the bifurcation process quantitatively. We will report the results for some typical attractors for at V_{ac} of 9, 34, 40, 50, 52, 56, 63, and 79 mV. The Poincaré maps for these driving amplitudes are all shown in Fig. 3 of Ref. 13 except for 50 mV. At 50 mV, the oscillation is frequency locked, and the Poincaré map consists of three points, similar to that for 40 mV.

III. CHARACTERIZATION OF ATTRACTORS

As mentioned in Sec. I, an attractor can be characterized quantitatively by a multifractal dimension D_q, where q is a continuous index. If the attractor is a point, line, or two-dimensional surface, then D_q is 0, 1, or 2, respectively, independently of the parameter q. However, if the attractor becomes more complicated, then its dimension can be a fractal number, whose actual value varies with q. In order to determine the fractal dimensions, the attractor is first covered with a grid of cubes. Then the frequency, with which a trajectory visits the various cubes covering the attractor, is calculated in the limit that the trajectory length goes to infinity.[3,4] This frequency is used to define various fractal dimensions such as the box-counting, information, or correlation dimensions.[2] As shown previously (cf. Fig. 3 of Ref. 13), the point density varies considerably in different parts of a chaotic attractor, indicating that some regions are much more frequently visited than others. Instead of using a grid of cubes, another equivalent method is to calculate the correlation between different points, a method proposed by Grassberger and Procaccia[14] for the case of $q=2$ and later generalized for the whole spectrum of dimensions D_q.[15] In this method the multifractal dimension D_q is defined as[14,15]

$$D_q=\frac{1}{q-1}\lim_{r\to 0}\frac{\ln[C_q(r)]}{\ln(r)}, \tag{1}$$

with the generalized correlation function $C_q(r)$ given by

$$C_q(r)=\frac{1}{N}\sum_{i=1}^{N}\left[\frac{1}{N}\sum_{j=1}^{N}\theta(r-|\vec{x}_i-\vec{x}_j|)\right]^{q-1}, \tag{2}$$

where N is the number of points, $\vec{x}_i$ is a point on the attractor, and θ is the Heaviside step function. In these equations, q can be any number. D_0 is known as the capacity dimension, D_1 as the information dimension, and D_2 as the correlation dimension. In fact, the multifractal dimension in this method is calculated through the generalized probability to find any two points of the attractor within a distance r, which is easier to compute in practice and gives smooth curves as compared to the grid method.[16]

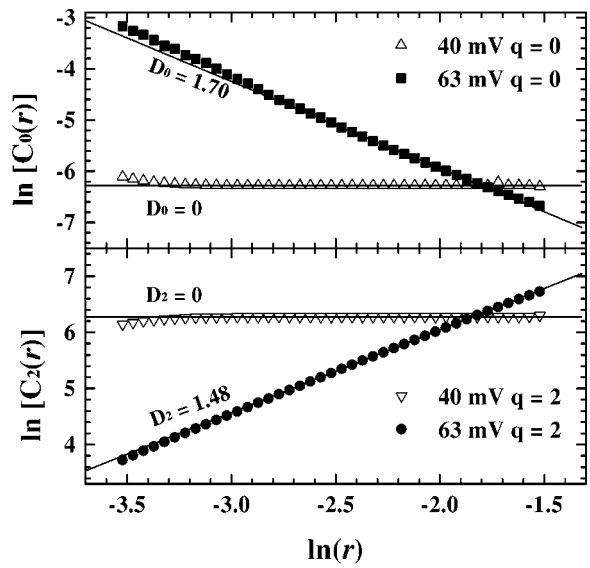

FIG. 1. $\ln[C_q(r)]$ vs $\ln(r)$ for $q=0$ and 2 at $V_{ac}=40$ and 63 mV, respectively. The solid lines are linear fits to the data points.

As q is varied, different regions of the attractor will determine D_q. D_∞ corresponds to the region where the points are mostly concentrated, while $D_{-\infty}$ is determined by the region where the points have the least probability to be found.[3] If D_q is a constant for all q, then the trajectory will visit different parts of the attractor with the same probability, i.e., the point density is uniform in the Poincaré map. This type of attractor is a so-called ''trivial'' attractor. When D_q changes with q reflecting a different point density among the attractor, it is denoted a ''nontrivial'' attractor. Therefore, the D_q curves can be used to determine the structure of the attractor. In the following, we will compute D_q curves from the experimentally obtained Poincaré maps shown in Fig. 3 of Ref. 13. We will then discuss the structure of the different attractors observed in the bifurcation route to chaos.

Figure 1 shows several $\ln[C_q(r)]$ vs $\ln(r)$ curves for 40 and 63 mV, which are calculated from the corresponding Poincaré maps. In this figure we only give examples for $q=0$ and 2. In these curves, a linear region is found for a certain range of r values. For smaller and larger r, the curves depart from a linear dependence due to the finite number of points in the Poincaré map and the finite size of the attractor. Although the limit $r\to 0$ is not very well fulfilled because in our experiment the number of the points in the Poincaré map is limited to 1600, the dimension of the attractors can still be estimated according to Eq. (1) by determining the slope of the linear region in Fig. 1. For 40 mV, the fractal dimensions D_0 and D_2 obtained in this way are approximately equal to zero, which is in agreement with a frequency-locked attractor. However, for the chaotic attractor at 63 mV, the dimensions $D_0=1.70$ and $D_2=1.48$ are fractal. They even vary with q.

Figure 2(a) shows the D_q vs q curves for the bifurcation

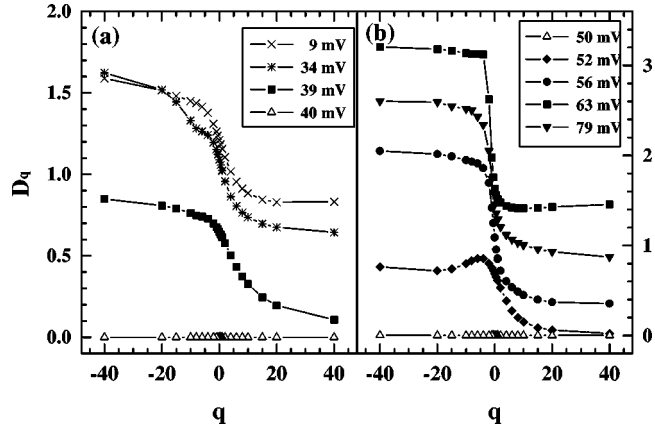

FIG. 2. Multifractal dimensions D_q vs q for the driving amplitudes V_{ac} from 9 to 40 mV (a) and from 50 to 79 mV (b) obtained from the corresponding Poincaré maps.

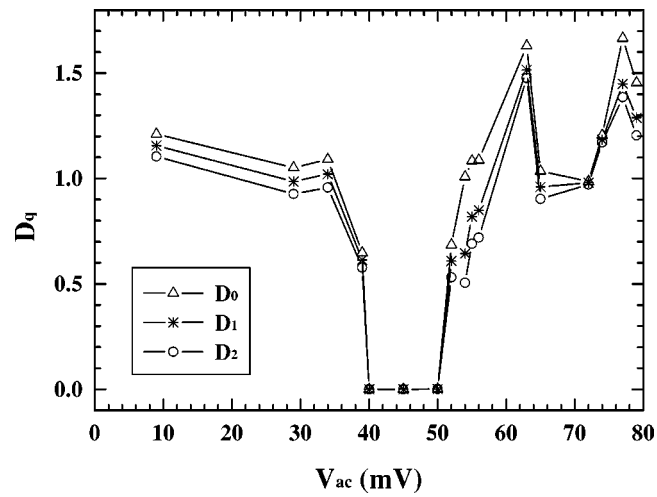

FIG. 3. Dimensions D_0, D_1, and D_2 as a function of the driving amplitude V_{ac}.

sequence from quasiperiodicity → synchronized chaos → frequency locking (1 to 40 mV). For the frequency-locked state at 40 mV, the multifractal dimension D_q is approximately equal to zero for all values of q in good agreement with the corresponding attractor of three isolated points. The attractor for frequency locking is therefore not strange. For quasiperiodicity, the Poincaré map should be a smooth loop.[17] Consequently, it is expected that a quasiperiodic attractor is one dimensional, i.e., D_q is equal to one for all values of q. However, the D_q curves for the quasiperiodic attractors in our experiments (between $V_{ac}=1$ and 37 mV) do not exhibit a horizontal line as shown for 9 and 34 mV in Fig. 2(a), although D_0 is close to one. $D_{-\infty}$ is about 1.6 and D_∞ approaches a finite value below one. The ''kneelike'' structure of D_q shown in Fig. 2(a) is typical for chaotic attractors with multifractal dimensions. We therefore conclude that the quasiperiodic attractors are nontrivial in our system. This can also be seen from the Poincaré maps in Fig. 3 of Ref. 13, since there appears to be in some parts a double-layer structure of the loop. Strange attractors are not always connected with chaos. The possibility of their emergence in nonchaotic systems has been predicted theoretically by Grebogi *et al.*,[18] and they have been experimentally observed in different physical systems, such as quasiperiodically forced systems[19] and an undriven gas-discharge plasma,[20] etc. Although there is no complete theory of such a behavior, usually the appearance of nonchaotic strange attractors is related to the intermittency induced by noise.[21] A possible mechanism of their emergence at the boundaries of mode-locked tongues is discussed in Ref. 22.

For synchronized chaos at 39 mV, the D_q curve also shows a ''kneelike'' structure. In contrast to the quasiperiodic regime, $D_{-\infty}$ is less than 1, while D_∞ approaches zero. This dependence indicates that the synchronized chaos observed at 38–39 mV exhibits the characteristics of both chaos and frequency locking. However, the attractor for synchronized chaos has a capacity dimension D_0 of about 0.65, which is less than that of the quasiperiodic attractors.

When V_{ac} increases beyond 50 mV, an explosive bifurcation sequence to chaos is observed as discussed in Sec. II. Figure 2(b) gives several multifractal dimensions for the attractors in this bifurcation regime. At 50 mV, the oscillations are periodic so that the dimension of the attractor is about zero. This is actually the case for all values of V_{ac} between 40 and 50 mV. For V_{ac} larger than 50 mV, the Poincaré maps expand into more complicated structures. At the same time, the D_q curves become nontrivial exhibiting the typical kneelike structure of strange attractors. With increasing V_{ac} from 50 to 56 mV, the height of the kneelike structure increases indicating that the attractor becomes more strange. The system enters a chaotic regime with higher complexity. Additionally, the strangeness for the chaotic attractors in this regime is more pronounced than for synchronized chaos at 39 mV and also for the quasiperiodic regime. This can be easily seen in Fig. 2(b), since the absolute values of D_q as well as the height of the kneelike structure are larger than in Fig. 2(a). A number of chaotic attractors with different multifractal dimensions are observed between 56 and 80 mV. Figure 2(b) gives examples for 56, 63, and 79 mV.

Figure 3 shows the dependence of D_0, D_1, and D_2 on V_{ac}. It is very clear from this graph that the multifractal dimension of the attractor is changing during the bifurcation sequence to chaos. The dimensions D_0, D_1, and D_2 do not change very much within the quasiperiodic regime. When the frequency-locked state is reached at 40 mV, all three fractal dimensions vanish. Between 50 and 63 mV, all three fractal dimensions increase abruptly reaching slightly different maximum values of about 1.5 at 60 mV. The abrupt increase of the fractal dimensions demonstrates that the bifurcation sequence from frequency locking to chaos is explosive. Between 51 and 80 mV, a number of strange attractors is observed in the chaotic regime with varying fractal dimensions. Since for random noise the capacity dimension D_0 is equal to 2 (for the two-dimensional phase space considered here), we conclude that the observed dimension D_0 between 51 and 80 mV with values of less than 2 clearly demonstrates the existence of deterministic low-dimensional chaos in our experiments.

IV. SUMMARY

We have calculated the multifractal dimension of the attractors in an experimentally observed bifurcation sequence

to chaos for a driven, weakly coupled semiconductor SL. These dimensions are obtained from Poincaré maps, which are constructed from real-time current traces and measure the strength of the multifractality of the corresponding attractor. The dependence of the fractal dimension on the driving voltage amplitude demonstrates quantitatively that the bifurcation sequence contains quasiperiodic, periodic, and chaotic attractors. Furthermore, the transition from frequency locking to chaos is explosive. In the quasiperiodic regime, the attractors may exhibit a certain degree of strangeness. Finally, the observed fractal dimensions unambiguously identify the existence of strange attractors and thereby confirm the deterministic nature of the observed chaotic oscillations.

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