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Calculation of the magnetic stray field of a uniaxial magnetic domain

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We present an analytic solution for the magnetic field of a bar-shaped permanent magnet. Assuming a constant magnetization, we derive expressions for the stray field in three dimensions. The analytic solutions can be readily applied to field calculation problems for magnetic force microscopy simulations without the need for finite element methods. © *2005 American Institute of Physics*. [DOI: 10.1063/1.1883308]

I. INTRODUCTION

Micro- and nanomagnetic systems often consist of patterned magnetic films of rectangular shape. Due to shape anisotropy, the small single domain magnets behave like uniaxial bar magnets. The analytic solution for the magnetic field of a ferromagnetic material of highly symmetrical shape, like a sphere or a cylinder, is a classical textbook problem of magnetostatics.[1,2] The magnetic fields of more complex geometries are derived numerically employing numerical micromagnetic solvers.[3] So far, analytic expressions for the demagnetization factor of uniaxial bar magnets were deduced,[4–6] and the interaction fields for a three-dimensional array of ferromagnetic cubes were calculated.[7] Unfortunately, no analytic expressions are given for the magnetic field of a bar magnet, which are especially useful for quantitative magnetic force microscopy (MFM) simulations, since they give access to the precise magnetic field values along with their derivatives.[8] They are the basis for fast and accurate computations of the force interaction.[9] Here we present an analytic solution for the magnetic field of a bar magnet, deduced from the Maxwell equations.

II. DERIVATION AND SOLUTION

The fundamental equations of magnetostatics are $\nabla\cdot\mathbf{B}=0$ and $\nabla\times\mathbf{H}=\mathbf{j}_f$, neglecting time derivatives, where $\mathbf{j}_f$ is the current density of the free charges. The magnetic flux density $\mathbf{B}$ and the magnetic field are linked by the material equation $\mathbf{B}=\mu_0\cdot(\mathbf{H}+\mathbf{M})$, where μ_0 and $\mathbf{M}$ are the permeability and the magnetization, respectively.

The geometry of the bar magnet, for which we assume a constant magnetization, is shown in Fig. 1. The magnetic material is defined within a volume $V=\{|x|\leqslant x_b, |y|\leqslant y_b, |z|\leqslant z_b\}$. Without loss of generality, the magnetization is chosen along the y axis: $\mathbf{M}=M_0\cdot\mathbf{e}_y$. For simplicity, $\mathbf{j}_f$ equals zero.

The magnetic field can be described introducing a scalar field Φ:

$$\mathbf{H}(\mathbf{r}) = -\nabla_{\mathbf{r}}\cdot\Phi(\mathbf{r}), \tag{1}$$

resulting in the Poisson equation for the scalar magnetostatic potential with the solution

$$\Phi(\mathbf{r}) = -\frac{1}{4\pi}\int\frac{\nabla_{\mathbf{r}_i}\cdot\mathbf{M}(\mathbf{r}_i)}{|\mathbf{r}-\mathbf{r}_i|}d^3\mathbf{r}_i. \tag{2}$$

A simpler expression for Φ is derived making use of Gauss' law and due to the localization of $\mathbf{M}$ in V:

$$\Phi(\mathbf{r}) = -\frac{1}{4\pi}\nabla_{\mathbf{r}}\cdot\int\frac{\mathbf{M}(\mathbf{r}_i)}{|\mathbf{r}-\mathbf{r}_i|}d^3\mathbf{r}_i. \tag{3}$$

The symmetry suggests to treat the problem in Cartesian coordinates $\mathbf{r}=(x,y,z)$. Inserting the magnetization in Eq. (3) and integrating over the cuboid yields

$$\Phi(x,y,z) = -\frac{M_0}{4\pi}\frac{\partial}{\partial y}\times\int_{-x_b}^{x_b}\int_{-y_b}^{y_b}\int_{-z_b}^{z_b}\frac{dx_i dy_i dz_i}{\sqrt{(x-x_i)^2+(y-y_i)^2+(z-z_i)^2}}. \tag{4}$$

By integrating Eq. (4) one obtains the analytic expressions for the magnetic field components of the bar magnet, given in Eqs. (5)–(7). A detailed deduction is presented in the Appendix. By choosing the center of the magnetized volume to be in the origin of the coordinate system, the components of the magnetic field can be expressed as a sum of terms with alternating sign. A displacement of the magnetized volume by $\mathbf{r}_a=(x_a,y_a,z_a)$ is achieved by transforming $\mathbf{r}\to\mathbf{r}+\mathbf{r}_a$. The

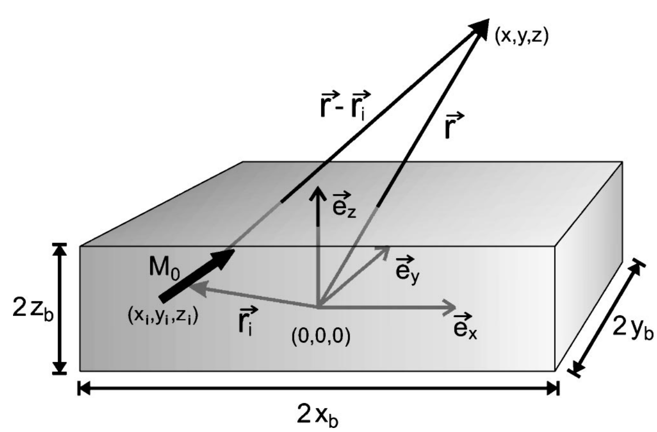

FIG. 1. Illustration of the integration geometry of the bar magnet. The magnetization M is along the y axis, the dimensions of the magnetized volume are $2x_b, 2y_b$, and $2z_b$.

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expressions are not defined in the corners of the cuboid, i.e., for $x=\pm x_c$, $y=\pm y_c$, and $z=\pm z_c$. The derivative of the magnetic field component H_y is not continuous at certain points as well, due to the appearance of the absolute value. It has to be noted that these problems are not relevant for the proposed applications.

III. DISCUSSION

The calculated magnetic field of the bar magnet with $x_b=4$, $y_b=10$, $z_b=2$ and a constant magnetization in the (positive) y direction is shown in Fig. 2. Three slices of the magnetic field in every dimension at the positions $x_c=2x_b$, $z_c=2z_b$, $y_c=-2y_b$ are presented as indicated in the sketch on top of the figure. Every cut is showing a two-dimensional vector plot of the magnetic in-plane components together with the image of the respective out-of-plane component. The in-plane field strength is visualized by the length of the arrows and the gray level of the vectors, and the underlying gray scale images correspond to the magnitude of the out-of-plane component. The maximum out-of-plane values are white or black, depending on the direction of the vector parallel or antiparallel to the base vectors of the coordinate system.

$$H_x(x,y,z)=\frac{M_0}{4\pi}\sum_{k,l,m=1}^{2}(-1)^{k+l+m}\ln\left\{z+(-1)^m z_b+\sqrt{[x+(-1)^k x_b]^2+[y+(-1)^l y_b]^2+[z+(-1)^m z_b]^2}\right\}, \quad (5)$$

$$H_y(x,y,z)=-\frac{M_0}{4\pi}\sum_{k,l,m=1}^{2}(-1)^{k+l+m}\frac{[y+(-1)^l y_b][x+(-1)^k x_b]}{|y+(-1)^l y_b||x+(-1)^k x_b|}\times\arctan\left\{\frac{|x+(-1)^k x_b|\cdot[z+(-1)^m z_b]}{|y+(-1)^l y_b|\cdot\sqrt{[x+(-1)^k x_b]^2+[y+(-1)^l y_b]^2+[z+(-1)^m z_b]^2}}\right\}, \quad (6)$$

$$H_z(x,y,z)=\frac{M_0}{4\pi}\sum_{k,l,m=1}^{2}(-1)^{k+l+m}\ln\left\{x+(-1)^k x_b+\sqrt{[x+(-1)^k x_b]^2+[y+(-1)^l y_b]^2+[z+(-1)^m z_b]^2}\right\} \quad (7)$$

The presented analytic solution applies to a sample with constant magnetization. For instance, this is the case for a system with remanent magnetization M_r close to saturation magnetization M_s. Our approach yields an error that increases with the difference $\Delta M=M_r-M_s$ between remanent magnetization and saturation magnetization. For small differences ΔM, the relative errors of the components of the magnetic stray field are approximately given by $\Delta M/M_s$.

For comparing the derived analytic expressions with numerical results, we calculated the magnetic flux density B of a bar magnet along a line $x=z$ on its surface $(y=y_b)$ as described in Ref. 10. The magnetization was $M_0=870$ kA/m and the dimensions of the permanent magnet 20 mm $\times$40 mm$\times$20 mm. Figure 3 shows the resultant magnetic flux densities obtained by a finite element method (from Ref. 10) and by employing Eqs. (5)–(7). Our analytic expressions do not allow the continuous computation of the magnetic flux density on the surface of the magnet, because of the undefined corner points.[11] Therefore, we performed the calculation in the near vicinity $(y=y_b+\varepsilon;\varepsilon=10^{-6}y_b)$ of the surface.

Furthermore, we compared the results obtained from the analytic expressions with a calculation performed with the bar magnet calculator from Integrated Engineering Software.[12] Using the same geometry as in Fig. 1 we obtained coinciding results (see Table I).

IV. SUMMARY

In summary, we present a derivation of the components of the magnetic field of a permanent magnet with cuboidal geometry. We compare the calculated field values with numerically obtained data. The analytic solution is the basis for quantitative MFM, as the magnetic stray field of the sample and their derivatives can be calculated fast and accurately.

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APPENDIX

By transforming $\partial/\partial y\to-\partial/\partial y_i$ in Eq. (4) and integrating with respect to y_i, we obtain the integrand in its integration limits

$$\Phi=-\frac{M_0}{4\pi}\int_{-x_b}^{x_b}\int_{-z_b}^{z_b}\left[\frac{1}{\sqrt{(x-x_i)^2+(y+y_b)^2+(z-z_i)^2}}-\frac{1}{\sqrt{(x-x_i)^2+(y-y_b)^2+(z-z_i)^2}}\right]dx_i dz_i. \quad (A1)$$

By taking Eq. (1) into account, we derive for H_x a sum of four terms due to the integration limits in the x and the y direction

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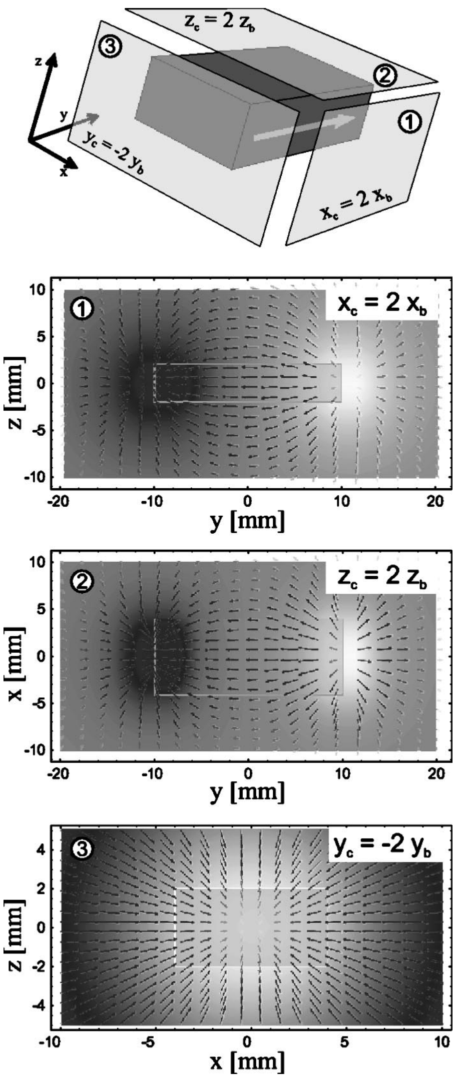

FIG. 2. Magnetic field of a bar magnet: the three plots are cuts through the three-dimensional vector field at the positions $x_c=2x_b$, $z_c=2z_b$, and $y_c=-2y_b$, respectively. These positions are indicated with respect to the bar magnet in the sketch on the top.

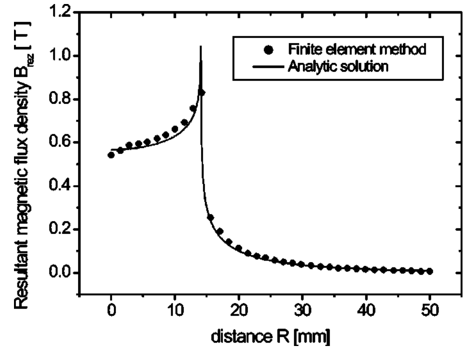

FIG. 3. Comparison of the resultant magnetic flux density along a line $x=z$ on the surface $(y=y_b)$ of a bar magnet. Dots: finite element method (cf. Ref. 10) and solid line: analytic expression.

$$H_x = -\frac{\partial}{\partial x}\Phi = \frac{M_0}{4\pi}\Big[I_{x+x_b}^{y+y_b} - I_{x-x_b}^{y+y_b} - I_{x+x_b}^{y-y_b} + I_{x-x_b}^{y-y_b}\Big], \tag{A2}$$

with

$$I_{x\pm x_b}^{y\pm y_b} = \int_{-z_b}^{z_b} \frac{dz_i}{\sqrt{(x\pm x_b)^2 + (y\pm y_b)^2 + (z-z_i)^2}}. \tag{A3}$$

Equation (A3) can be integrated with respect to z_i resulting in Eq. (5). The structure of the expressions H_z and H_x is symmetric in x and z due to the choice of the magnetization in the y direction. The deduction of the expression for H_z is achieved by changing the integration order compared to the integration leading to H_x.

For the H_y component the expression remains complicated. With Eqs. (1) and (A1) we obtain

TABLE I. Comparison of the components of the magnetic stray field calculated from Eqs. (5)–(7) (AS) for the cuboid $(x_b,y_b,z_b)=(4,10,2)$ with a remanent magnetization of 796 kA/m at position $\mathbf{r}=(x,y,z)$ with the values obtained by a finite element calculation (FEM) available from Ref. 12.

x	y	z	B_x(mT)	B_y(mT)	B_z(mT)	
$1.1x_b$	$1.1y_b$	$1.1z_b$	112.814	59.737	86.680	AS
			112.811	59.723	86.679	FEM
$2x_b$	$2y_b$	$2z_b$	7.268	8.330	3.892	AS
			7.268	8.330	3.892	FEM
$10x_b$	$10y_b$	$10z_b$	39.413×10^{-3}	58.473×10^{-3}	19.740×10^{-3}	AS
			39.413×10^{-3}	58.473×10^{-3}	19.734×10^{-3}	FEM
$1.1x_b$	$1.1y_b$	0	180.916	114.796	0	AS
			180.916	114.765	0	FEM
$2x_b$	$2y_b$	0	8.400	9.935	0	AS
			8.400	9.935	0	FEM
$10x_b$	$10y_b$	0	42.985×10^{-3}	65.197×10^{-3}	0	AS
			42.985×10^{-3}	65.197×10^{-3}	0	FEM

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$$H_y(x,y,z) = -\frac{\partial}{\partial y}\Phi(x,y,z) = \frac{M_0}{4\pi}\int_{x_b}^{x_b}\int_{-z_b}^{z_b}\left\{\frac{y+y_b}{\left[(x-x_i)^2+(y+y_b)^2+(z-z_i)^2\right]^{3/2}} - \frac{y-y_b}{\left[(x-x_i)^2+(y-y_b)^2+(z-z_i)^2\right]^{\frac{3}{2}}}\right\}dx_i dz_i$$

$$= \frac{M_0}{4\pi}\int_{-z_b}^{z_b}\cdot\left\{\frac{y+y_b}{(y+y_b)^2+(z-z_i)^2}\left[\frac{x+x_b}{\sqrt{(x+x_b)^2+(y+y_b)^2+(z-z_i)^2}} - \frac{x-x_b}{\sqrt{(x-x_b)^2+(y+y_b)^2+(z-z_i)^2}}\right]\right.$$

$$\left. - \frac{y-y_b}{(y-y_b)^2+(z-z_i)^2}\left[\frac{x+x_b}{\sqrt{(x+x_b)^2+(y-y_b)^2+(z-z_i)^2}} - \frac{x-x_b}{\sqrt{(x-x_b)^2+(y-y_b)^2+(z-z_i)^2}}\right]\right\}dz_i.$$

This integral is still analytic and the result is given in Eq. (6).

[1]W. Nolting, *Grundkurs: Theoretische Physik 3. Elektrodynamik* (Verlag Zimmermann-Neufang, Germany, 1990); J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962).
[2]A. Hubert and R. Schäfer, *Magnetic Domains* (Springer, Germany, 1998).
[3]LLG Simulator, http://llgmicro.home.mindspring.com/; OOMMF, http://math.nist.gov/oommf/; MicroMagus, http://www.micromagus.de/.
[4]R. I. Joseph and E. Schlömann, J. Appl. Phys. **36**, 1579 (1965).
[5]G. Zheng, M. Pardavi-Horvath, X. Huang, B. Keszei, and J. Vandlik, J. Appl. Phys. **79**, 5742 (1996).
[6]A. Aharoni, J. Appl. Phys. **83**, 3432 (1998).
[7]M. E. Schabes and A. Aharoni, IEEE Trans. Magn. **MAG-23**, 3882 (1987).
[8]Th. Kebe and A. Carl, J. Appl. Phys. **95**, 775 (2004).
[9]R. Engel-Herbert *et al.* (unpublished).
[10]V. Čingoski and H. Yamashita, J. Appl. Phys. **81**, 4088 (1997).
[11]A. Thiaville, D. Tomas, and J. Miltat, Phys. Status Solidi A **170**, 125 (1998).
[12]http://www.integratedsoft.com/papers/benchmark/bar_magnet/calculator.asp

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