Strong reflection and periodic resonant transmission of helical edge states in topological-insulator stub-like resonators

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Strong reflection and periodic resonant transmission of helical edge states in topological-insulator stub-like resonators

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The helical edge states of two-dimensional topological insulators (TIs) experience appreciable quantum mechanical scattering in narrow channels when the width changes abruptly. The interference of the geometry scattering in narrow-wide-narrow waveguide structures is shown to give rise to the strong suppression of transmission when the incident energy is barely above the propagation threshold. Periodic resonant transmission takes place in this high reflection regime while the length of the wide section is varied. The resonance condition is governed by the transverse confinement in the wide section, where the form of quantization is manifested to differ for the two orthogonal directions. The confined energy levels in TI quantum dots are derived based on this observation. In addition, the off-diagonal spin-orbit term is found to produce an anomalous resonance state, which merges with the bottom ordinary resonance state to annihilate. © 2015 AIP Publishing LLC.

I. INTRODUCTION

The most prominent transport feature in topological insulators (TIs) is the absence of backscattering from nonmagnetic impurities. The edge states of a two-dimensional (2D) TI that form a Dirac cone in the energy range of the bulk band gap possess a helical configuration between the spin and momentum. That is, the spin is locked at a right-angle to the momentum as a consequence of strong spin-orbit coupling. As nonmagnetic impurities do not flip spin, the scattering between the counter-propagating edge states along a boundary is blocked by the opposite spin orientation. The scattering is allowed only to the backward moving state at the opposite boundary of a channel, where the probability of such a scattering is, in general, negligibly small due to the spatial separation between the two boundaries. The insignificance of backscattering can give rise to a high carrier velocity, which would be attractive for device applications. On the other hand, it implies a fundamental difficulty for externally controlling the transport properties, i.e., changing the conductance of the system, in order to accomplish functional device operations. Shifting the Fermi level by gating as in conventional electronic devices, for instance, is not very useful for the Dirac materials due to the absence of the band gap.

Numerical simulations have confirmed that the protection from scattering when a short-range disorder is imposed in the system is robust. The conductance of a 2D TI channel remains to be almost the quantized values expected for ballistic transport even in the presence of a moderate disorder unless the disorder is made significantly strong. In contrast, scattering occurs easily when the width of a channel is abruptly altered. Periodic modulations appear in the transmission through a narrow segment terminated by wide leads when the constriction size or the Fermi level is varied, evidencing multiple reflections between the narrow-wide junctions. As a matter of fact, one scarcely recognizes

II. MODEL

In Fig. 1, a narrow-wide-narrow (NWN) quantum waveguide structure is illustrated. The transport properties of an
ordinary 2D electron gas (2DEG) in the waveguide structure were investigated by Sols et al.,\textsuperscript{10,11} for the case of the length $L$ of the wide section being equal to the width $W_a$ of the narrow channels. There, nearly complete reflection was found to occur whenever resonance conditions were satisfied while the stub length $W_w$ was varied. The transmission modulation is a manifestation of the waveguide structure as a quantum stub resonator, where the reflection originates from the transmission resonance through quasi-zero-dimensional (quasi-0D) states induced in the stub segment. We examine whether the wide segment in the NWN structures operates also as a resonator for TIs.

The quantum transmission properties for the helical edge states are evaluated based on the effective four-band Hamiltonian,\textsuperscript{6,14} where a lattice parameter of $a = 5 \text{ nm}$ was used. The conductance of the system is related to the transmission probability $T = \text{Tr} \, \rho$, by the Landauer formula as $G = (e^2/h)T$. The transmission matrix $\rho$ was calculated using the lattice Green’s function method.\textsuperscript{6,15,16}

### III. DISPERSION IN WIRES

In Fig. 2(a), the energy dispersion in narrow strips of the 2D TI is shown for the widths of $W = 0.35, 0.39,$ and $0.47 \mu m$. As the spin-orbit coupling described by the parameters $R_0$ and $\Delta$ in Eq. (1) is taken into account, the spin degeneracy is lifted. It ought to be pointed out that the magnitude of the spin-orbit splitting depends exponentially on the channel width for the TEs.\textsuperscript{17} The channel velocity in this circumstance is no longer proportional to the wavenumber $k$. The positive- and negative-velocity branches are indicated by the solid and dotted curves, respectively. The arrows illustrate the direction of propagation given by the channel velocity obtained as the expectation value of the velocity operator $\hat{x} = (i/h)[H, \hat{x}]$ for the case of $W = 0.39 \mu m$.

It is evident in Fig. 2(a) that an energy gap opens at the Dirac point as a consequence of the bonding of wavefunctions at two boundaries of the channels. (The Dirac point energy is $E_D = -MD/B$ for the Hamiltonian in Eq. (1).) As shown by the dotted curve in Fig. 2(b), the hybridization energy gap $E_g$ shrinks almost exponentially as the channel width increases in the absence of the spin-orbit coupling ($\Delta = R_0 = 0$).\textsuperscript{18} In the presence of the spin-orbit coupling, $E_g$ becomes zero at certain channel widths as the overlap of the wavefunctions can vanish.\textsuperscript{9,19} For the TI HgTe quantum well, the gap disappearance occurs at $W \approx 390 \text{ nm}$, as shown by the solid curve. With increasing the strength of the spin-orbit coupling, the widths for the gap disappearance shift to be smaller, as one finds for the case shown by the dashed curve. There, one also notices that the energy gap disappears regularly as the width increases.

### IV. TRANSMISSION RESONANCE IN NWN STRUCTURES

Typical transmission characteristics of the helical edge states in the NWN structure when $L$ is varied are shown in Fig. 3(a) for a case of $W_n = 390 \text{ nm}$ and $W_w = 445 \text{ nm}$. Although the transmission is no longer perfect as a consequence of the nonuniform channel width ($W_w/W_n = 1.141$), the amplitude of the oscillatory modulation is significantly small. For this reason, the reflection probability $R = 2 - T$...
was plotted instead of \( T \) for convenience. (Note that \( 0 \leq T \leq 2 \) due to the spin freedom.) The circles in Figs. 3(b) and 3(c) show the dependencies of, respectively, the period \( \Delta L \) and the amplitude (peak-to-valley difference) \( \Delta R \) of the oscillation on the energy difference \( \delta E \) between the Fermi level \( E_F \) and the propagation threshold energy in the narrow leads in the NWN structure (7.463573 meV).

As we have illustrated in Fig. 1, the abrupt width changes in the NWN structure cause backscattering, as indicated by the dotted lines. Localized states are thus formed in the wide section, giving rise to quantum interference effects. The behaviors of the oscillatory modulation can be understood by a one-dimensional model, where the wavenumber \( k \) is assumed to be altered to \( q \) in an embedded segment having a length of \( L \) by an effective potential representing the width variation. The wavenumber discontinuity leads to a relationship \( R = \frac{2}{q} \left[ (k - q)/(k + q) \right] \left[ 1 - \cos(2qL) \right] \) when \( k \approx q \). When the incident energy increases, the ratio \( kl/q \) approaches unity, and so the oscillation amplitude decreases. The dependence in Fig. 3(c) is approximately described by a power law \( \Delta R \propto \delta E^{-1.8} \) as indicated by the solid line in the low energy regime. A strong deviation occurs, however, for high energies, presumably due to the influence of the bulk conduction band states. (The threshold for the occupation of the lowest bulk conduction band state in a wire with width \( W_w = 445 \) nm is 10.124 meV, which corresponds to \( \delta E = 2.66 \) meV.) On the other hand, \( \Delta L \) is inversely proportional to \( \delta E \) for the entire energy range reflecting the linear dispersion of the Dirac cone \( \delta E \approx A \left[ 1 - (B/D)^2 \right]^{1/2} q \), as shown by the solid line in Fig. 3(b). (The parabolicity of dispersion in the vicinity of the hybridization energy gap is negligible in Fig. 3(b) as \( \delta E \gg E_g \).

The solid curve in Fig. 4(a) shows \( T \) when \( E_F \) is further lowered to be barely above the propagation threshold. One finds that the transmission is strongly suppressed over, unexpectedly, the whole range of \( L \). The minimum transmission decreases rapidly as \( \delta E \) becomes smaller. Narrow peaks that evolve from the sinusoidal oscillation in Fig. 3(a) emerge periodically in this strong reflection regime. Notice that the resonance period is nearly independent of \( \delta E \), indicating deviation from the \( \Delta L \propto \delta E^{-1} \) behavior in Fig. 3(b). In fact, the shape of the transmission curve changes from being sinusoidal to narrow peaks at about the same value of \( \delta E \) for the transition from the \( \Delta L \propto \delta E^{-1} \) behavior to the independence of the peak positions on \( \delta E \). The manner of the resonance in the transmission demonstrates a fundamental difference in the performance of the quantum resonator between TIs and conventional 2DEGs. For the latter, nearly complete reflection is realized by the transmission resonance through quasi-bound states confined in the wide section.\(^{10,11,21}\) The

![FIG. 3. (a) Variation of reflection probability \( R \) when resonator length \( L \) is changed. The widths of the narrow and wide sections are \( W_w = 445 \) nm, respectively. \( W_w \approx W_n = 390 \) nm. The Fermi energy is 7.49, 7.5, and 1.41 meV for the solid line, respectively. The period \( \Delta L \) and the amplitude \( \Delta R \) of the oscillatory modulations are plotted in (b) and (c), respectively. \( \delta E \) is the excess energy at the Fermi level with respect to the propagation threshold energy in the narrow leads (7.463573 meV). The solid lines in (b) and (c) show behaviors \( \propto \delta E^{-1} \) and \( \propto \delta E^{-1.8} \), respectively. The data points corresponding to the three curves in (a) are indicated by the arrows in (c).](image1)

![FIG. 4. (a) Dependence of transmission probability \( T \) on length \( L \) of wide section in strong reflection regime. The incident energy \( \delta E \) was chosen to be 1.6 meV for the solid curves from top to bottom, respectively. The widths of the narrow and wide sections are \( W_n = 250 \) nm and \( W_w = 500 \) nm, respectively. The modulation period \( \Delta L \) and the length \( L_1 \) for the first transmission peak are defined as shown in the panel. The value of the spin-orbit parameter \( \Delta \) was 1.6 meV for the dotted curves. \( \Delta \) was increased to 5.1 meV with \( W_w = 170 \) nm, \( W_w = 230 \) nm, and \( \delta E = 10^{-4} \) meV. The small-\( L \) regime is shown with an expanded scale by the upper curve. (b) Dependencies of \( \Delta L \) and \( L_1 \) on \( W_w \) for \( W_n = 125 \) and 250 nm. The open and filled circles show \( \Delta L \) and \( L_1 \), respectively. The predictions of Eq. (2) are shown by the solid curves.](image2)
resonance takes place for TIs, in contrast, in the form of enhanced transmission.

A characteristic of the interference phenomenon of the helical edge states becomes apparent when $W_w$ is varied. The open circles in Fig. 4(b) show the variation of $\Delta L$ with $W_w$. Here, $\Delta L$ was evaluated in the limit of $\delta E \to 0$. The transmission is hardly affected by a change in $W_w$, as indicated by the approximate independence of $\Delta L$ on the ratio $W_w/W_n$. We have identified that $\Delta L$ is given by the two wavenumbers $k_1$ and $k_2$, which emerge as a consequence of the lifting of the spin degeneracy, in a channel having a width of $W_w$ as

$$\Delta L = \frac{2\pi}{k_1(E_F, W_w) + k_2(E_F, W_w)},$$

as shown by the solid curves. Note that $k_1k_2 < 0$ for the bottom energy segment of the dispersion in the presence of the spin-orbit coupling. ($k_1 = k_2$ when the spin-orbit terms are ignored.) The increase of $\Delta L$ when $W_w \approx W_n$ in Fig. 4(b) originates from the closeness of the hybridization energy gap edge in the wide section to the Fermi level. The values of $k_1$ and $k_2$ are thereby affected by the parabolic shape of the dispersion curve when $W_w$ is small.

In order to reveal the quantum interference process that is responsible for the transmission modulation, we plot in Fig. 5 the amplitude distribution of the local density of states

$$\rho(r) = -\frac{1}{\pi} \text{Im} G^+(r, r; E_F).$$

Here, $G^+(E) = (E - H + i\epsilon)^{-1}$ is the retarded Green’s function. The length of the wide section was set to the value for which the transmission exhibits the second peak when $L$ is increased from zero. (The third peak if we include the trivial peak at $L = 0$.) One finds that the wavefunction is localized along the boundary of the cavity section, as expected for the edge states. (The amplitude in Fig. 5 is plotted using the logarithmic scale. The wavefunction is seen to decay exponentially with increasing the distance from the boundary.) When the resonant transmission takes place, a standing wave pattern is found at the upper and lower boundaries of the wide section. The number of the nodes in the pattern changes by one for the successive resonant transmissions. The amplitude distributions associated with the $|e\rangle$ ($|h\rangle$) and $|e\rangle$ ($|h\rangle$) components are identical. In the interior of the wide section, minima in the pattern exhibited by $|e\rangle$ and $|h\rangle$ are approximately replaced with saddle points in the pattern for $|h\rangle$ and $|e\rangle$, respectively. In approaching the boundary of the cavity, the correspondence of the peaks and nodes between $|e\rangle$ and $|h\rangle$ becomes complicated, i.e., the wavefunction is not separable as the horizontal and vertical components even in the areas far away from the vertical boundary of the cavity. (Notice that the images in Figs. 5(a) and 5(b) are squeezed horizontally by a factor of ~5.)

In analogy to the stub resonator for 2DEGs, one may anticipate the transmission to modulate periodically as $W_w$ changes by amounts corresponding to integer multiples of a half of the wavelength at the Fermi level. As $\Delta L$ is considerably larger than $W_n$ for most of the cases shown in Fig. 4, it is not clear if such a modulation, indeed, occurs for TIs. Nevertheless, one recognizes in Figs. 5(c) and 5(d) that the edge states propagate along the vertical boundary of the waveguide, similar to the propagation along the horizontal boundary. The phase shift associated with the vertical propagation should, therefore, contribute to the interference effect. In Fig. 4, we plot the length $L_1$ of the wide section for the first transmission peak by the filled circles. While $L_1 \approx \Delta L$ for most of the cases, $L_1$ becomes considerably smaller than $\Delta L$ when $W_w/W_n$ is large for $W_n = 125$ nm. The deviation evidences that the vertical propagation cannot be ignored when $W_w$ is comparable to or larger than $\Delta L$, i.e., $\Delta L$ is anticipated to be identical to $L_1 + W_w - W_n$. We emphasize, however, that $L_1 = W_n$ is suggested in the limit of infinitely large $W_w$. The actual interference path when $W_w > \Delta L$ is more complicated than being simply along the boundary of the waveguide.

The relationship in Eq. (2) has implication for the formation of quasi-0D states in TI quantum dots. The narrowness of the transmission peaks in Fig. 4(a) suggests that the periodic modulation is due to resonances through quasi-bound states. For the conventional quantum stub resonator of 2DEGs, higher-lying quasi-0D levels fall below the Fermi

![Image](https://via.placeholder.com/150)

**FIG. 5.** Distribution of local density of states $\rho(r)$ for $W_w = 150$ nm, $W_w/W_n = 1.8$, and $\delta E < 10^{-7}$ meV. The transmission exhibits the second peak for the chosen length of $L/W_w = 17.63$. (a) and (c) correspond to $|e\rangle$ and (b) and (d) correspond to $|h\rangle$. The logarithm of the amplitude is plotted using gray scales. The amplitude in (a) is about an order of magnitude smaller than that in (b). The distributions around the narrow-wide junction in (a) and (b) are shown with 1-to-1 aspect ratio in (c) and (d), respectively.
level as \(W_w\) increases, giving rise to successive transmission resonances. In the case of the helical edge states, confinement in a finite width yields merely the hybridization of the wavefunctions and no formation of discrete energy levels, see Fig. 2. In this sense, changing the stub length \(W_w\) does not necessarily result in modulating the transmission properties as higher-lying states do not exist for the case of the TI quantum resonators. In contrast to the quantized energy levels of a 2DEG for the rectangular quantum dot \(E_{n,l} = (r^2/2m)((n\pi/k)^2 + ((n/L)^2)^2 \right)\) with \(n\) and \(l\) being integers, the apparent quantization rule for the bound states is \(kL_{\text{loop}} = 2\pi\), where \(L_{\text{loop}}\) is the length of the loop defined as the periphery of the TI quantum dot.\(^{22-24}\) The energy levels of the bound states are thus approximately estimated to be

\[
E_i = A \sqrt{1 - \left(\frac{D}{B}\right)^2} \frac{I\pi}{W_w + L} + E_D. \tag{4}
\]

Equation (2) indicates that the width discontinuity scatters an edge state at one boundary into the edge state having the different \(|\psi|\)-value at the opposite boundary. The quantization rule when the spin-orbit splitting is included is hence suggested to be

\[
k_1(W_w)L + k_2(W_w)L + k_1(L)W_w + k_2(L)W_w = 2I\pi, \tag{5}
\]

where \(k_1\) and \(k_2\) are evaluated in a wire having a width of \(W_w\) or \(L\). The two wavenumbers contribute equally in an individual state instead of producing each own bound states. We emphasize that the role of the two orthogonal directions on the quantization rule differs, i.e., one direction determines the value of \(k\) and the other direction sets the length of propagation. This may be the origin for the sensitivity of the transmission of helical edge states to a geometrical bending in the channel.\(^{28}\)

The waveguide structure was assumed to be defined with rectangular bends and corners for simplicity. These sharp geometries are likely to be rounded to a certain extent in real semiconductor devices produced using microfabrication technologies. As examined in Ref. 9, gradual widening of a channel reduces the geometrical reflection for the helical edge states, similar to the effect in tapered waveguides of 2DEGs.\(^{29}\) The modification is quantitative and it is rather difficult to nearly completely suppress the quantum-mechanical scattering. The resonance features are thus expected to remain qualitatively unchanged regardless of such details in the waveguide geometry. We note that the influence of bulk disorder is also anticipated to be insignificant given the protection for the helical edge states from nonmagnetic impurity scattering, unless the disorder is significantly large as demonstrated for a system described by the same Hamiltonian in Ref. 6.

It would be interesting to examine the transport properties in NWN waveguide structures made of graphene considering its massless Dirac fermions that possess linear energy dispersion relation and helicity. Nevertheless, the transport in graphene is mediated by 2D bulk states instead of edge-localized states. We expect the transmission resonances in graphene nanoribbons to exhibit characteristics resembling those for the 2DEG waveguides.

**V. EVANESCENT TRANSMISSION IN WIDE SECTION**

If the off-diagonal spin-orbit terms are taken into consideration, \(E_g\) does not shrink monotonically when the channel width increases. This peculiarity for the helical edge states provides a possibility that the wide section in the NWN structure acts as a potential barrier with respect to the incident state in the narrow leads. Such a situation was, in fact, intended for the case shown in Fig. 3 as \(W_n\) was chosen to be the width for vanishing \(E_g\) and \(E_g\) exhibits a maximum at the width set for \(W_w\). We were, however, unable to extract indication of blocked transmission as the length scale for the transmission modulation became too large for our computational resource when \(\delta E\) was reduced to be small enough for experiencing the potential barrier. In the remainder, therefore, we strengthen the spin-orbit coupling to \(\Delta = 5.1\) meV to raise the barrier height.

An example of the evanescent transmission in the wide section is shown by the dotted curve in Fig. 4(a). The transmission, indeed, decreases exponentially with increasing the length of the wide section. To be precise, a rapid decay in the regime \(0 < L < W_n\) is followed by a slow decay. The two decay lengths are presumed to correspond to the two solutions of the secular equation for TIs.\(^{2,18}\) In addition, it is seen that an oscillation-like behavior can occur in the initial stage \(L \ll W_n\). When the parameter values are changed, following observations are made (not shown here). The decay length \(\zeta\) for the slow decay is almost independent of \(\delta E\) when \(\delta E\) is small as the Fermi level is well-below the top of the potential barrier created in the wide section. The magnitude of the rapid decay component decreases even in this

![FIG. 6. Period \(\Delta L\) for resonant transmission (circles) and decay length \(\xi\) for evanescent transmission (triangles) in wide section for \(\Delta = 5.1\) meV. The width \(W_w\) of the narrow leads was varied when the width of the wide section was fixed to be \(W_n = 230\) nm. \(\Delta L\) and \(\xi\) are practically independent of \(E_g\) as \(\delta E\) was chosen to be small (<10\(^{-7}\) meV). The solid curves show predictions of Eq. (2). The hybridization energy gap vanishes at the width of \(W_w\) indicated by the vertical dashed bar. The inset shows the transmission curve for the case marked by the dotted circle (\(W_w = 175\) nm). The transmission probability \(T\) is plotted when the length \(L\) of the wide section is varied. The dotted horizontal bar indicates the regime where the \(\Delta L\) periodicity is observed. An additional oscillatory behavior is present in the short-\(L\) regime marked by the horizontal solid bar. The two peaks marked by the arrows merge with each other and then annihilate if \(\Delta\) is increased, see Fig. 7.]

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circumstance when $\delta E$ is increased. Varying $W_n$ with a fixed value of $\delta E$ is, on the contrary, effective to alter $\zeta$ as the position of the Fermi level with respect to the barrier height changes significantly. As one would expect, the decay becomes rapid when $W_n$ approaches the value for vanishing energy gap, as shown by the triangles in Fig. 6. Here, the vertical dashed bar marks the location for the energy gap disappearance.

Due to the non-monotonic dependence of $E_g$ on the channel width, $\Delta L$ exhibits a rapid variation when $W_n$ changes around a gap vanishing value, see the circles in Fig. 6. The prediction of Eq. (2) shown by the solid curve is again in excellent agreement with the numerical results. In the inset of Fig. 6, we show the transmission curve when $W_n$ is just wider than the width for the evanescent transmission in the wide section. We point out that additional resonances emerge in the small $L$ regime marked by the horizontal solid bar. The peak separation seems to increase nearly exponentially with $L$. Here, the region of the ordinary resonances with the $\Delta L$ separation is indicated by the dotted bar.

For further investigating the characteristics of the anomalous peaks, we have varied the value of $\Delta$ when the waveguide widths are fixed to be $W_n = 175$ nm and $W_w = 230$ nm. The lengths $L_i$ for the $i$th peaks are plotted in Fig. 7 by symbols. The filled symbols indicate the ordinary resonances that are separated by $\Delta L$ from each other. As the peak shown by the filled triangles appears at small values of $L$, the peak position shown by the filled circles is nearly identical with $\Delta L$ given by Eq. (2), which is shown by the solid curve, for the entire range of $\Delta$. The anomalous peaks are indicated by the open circles. The grayed segments represent the ranges of the evanescent transmission in the wide section. The anomalous peaks are present only on the large-$\Delta$ side of the evanescent transmission region, in correspondence with the observation of the anomalous peaks on the small-$W_n/W_w$ side of the evanescent transmission region in Fig. 6. Remarkably, the state responsible for the anomalous peak merges with the bottom resonance state of the ordinary transmission peaks. (The pair is indicated by the arrows in the inset of Fig. 6.) The merged state then eventually annihilates. This behavior is repeated for the successive energy gap disappearances. Presently, the origin of the anomalous peaks, which may be related to the initial oscillatory behavior in the evanescent transmission in Fig. 4(a), is not understood.

The transmission through a square potential barrier in graphene is known to lead to the Klein paradox, where the electron tunneling through the barrier of arbitrary height and thickness is unimpeded. This exotic phenomenon originally predicted for relativistic particles originates from the fact that an incident electron propagates in the barrier as a positron (a hole in the graphene case) with no decay. In our situation, the effective potential barrier is created by width variations, and so the Dirac point energy is unchanged throughout the waveguide. As no transition between the electron and hole branches of the Dirac cone occurs during the transmission through the waveguide, the possibility that the anomalous behavior is related to the Klein problem is ruled out.

**VI. CONCLUSIONS**

In conclusion, we have evaluated the feasibility of NWN quantum waveguide structures serving as a quantum resonator for the helical edge states of TIs. Due to the protection of the helical edge states from backscattering, the transmission modulation as a consequence of quantum interference effects has been observed to be considerably weak. Nevertheless, strong reflection over broad parameter ranges occurs when the incident energy is lowered to be barely above the propagation threshold. We have established the interference condition in the presence of the off-diagonal spin-orbit terms, where the edge propagation has been recognized to play an essential role. The guideline for designing the quantum resonator is summarized as follows. The period of the modulation is determined by the wavenumbers in the wide section. If a short modulation period is required, the Fermi energy needs to be raised. As the resonance takes place in the manner of increasing the transmission, the modulation amplitude, however, decreases in such a circumstance. Reducing the width of the narrow leads allows us to accomplish short $\Delta L$ while maintaining large modulation amplitude by keeping the incident energy in the leads small. Our findings have provided insights in understanding the confined states in TI quantum dots. Additionally, the possible disappearance of the hybridization energy gap when the off-diagonal spin-orbit terms are taken into account has been demonstrated to give rise to a number of unconventional transport phenomena.

\[^{6}\text{M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. 82, 3045 (2010).}\]
\[^{7}\text{X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. 83, 1057 (2011).}\]

**FIG. 7.** Lengths $L_i$ for $i$th transmission peaks when $\Delta$ is varied. The widths of the NWN structure are $W_w = 175$ nm and $W_n = 230$ nm. The incident energy $\delta E$ was set to be small ($<10^{-7}$ meV). The filled symbols indicate the ordinary resonance peaks that exhibit the $\Delta L$ periodicity. For clarity, additional transmission peaks that occur at larger $L$ than for the filled circles are not included. The open symbols show the anomalous peaks. The solid curve shows $\Delta L$ given by Eq. (2). Evanescent transmission takes place in the wide section, i.e., the hybridization energy gap in the wide section becomes larger than that in the narrow leads, in the grayed ranges of $\Delta$. Note that the horizontal axis is divided into four separate regions with different scales to highlight the sections that contain the anomalous peaks. A scale change occurs at the value marked by the dotted line.
While nearly complete transmission is anticipated for the peaks associated with transmission resonances, we note that the peak amplitude becomes considerably smaller than 2 when $D_L$ is not large, for instance, for $W_n = 125$ nm in Fig. 4(b).